# CSM 61B Spring 2019

# Asymptotics & Disjoint Sets

Mentoring 5: February 25, 2019

## 1 Analysis of Algorithms

The **running time** of a program can be modeled by the number of instructions executed by the computer. To simplify things, suppose arithmetic operators (`+`, `-`, `*`, `/`), logical operators (`&&`, `||`, `!`), comparison (`==`, `<`, `>`), assignment, field access, array indexing, and so forth take 1 unit of time. `(6 + 3 * 8) / 3` would take 3 units of time, one for each arithmetic operator.

While this measure is fine for simple operations, many problems in computer science depend on the size of the input: `fib(3)` executes almost instantly, but `fib(10000)` will take much longer to compute.

**Asymptotic analysis** is a method of describing the run-time of an algorithm *with respect* to the size of its input. We can now say,

> The run-time of `fib` is, at most, within a factor of $2^N$ where $N$ is the size of the input number.

Or, in formal notation, `fib(n)` $\in O(2^N)$.

1.1 Define, in your own words, each of the following asymptotic notation.

(a) $O$

'Big-O' notation gives an *upper* bound on the runtime of a function as the size of the input approaches infinity.

(b) $\Omega$

'Big-Omega' notation gives a *lower* bound on the runtime of a function as the size of the input approaches infinity.

(c) $\Theta$

'Big-Theta' notation gives a *tight* bound on the runtime of a function as the size of the input approaches infinity.

1.2 Give a tight asymptotic runtime bound for `containsZero` as a function of $N$, the size of the input array in the *best case, worst case, and overall.*

```
public static boolean containsZero(int[] array) {
    for (int value : array) {
        if (value == 0) {
            return true;
        }
    }
    return false;
}
```

$\Theta(1)$ in the best case, $\Theta(N)$ in the worst case, and $\Omega(1), O(N)$ overall.

Asymptotic analysis is always concerned with what happens as our input grows to infinity. In this case, we'd like to describe the order of growth of `containsZero` as a function of the *size* of the input array, but that doesn't say anything about the contents of the input array.

We could find a zero at the beginning of the array: this is the best case as we can terminate in $\Theta(1)$. Or there could be no zeroes in the array at all: this is the worst case, which terminates $\Theta(N)$. We can then describe the overall runtime of the function taking into account all possible cases from spanning from best to worst: hence, the overall runtime is $\Omega(1), O(N)$.

# 2 Something Fishy

Give a tight asymptotic runtime bound for each of the following functions. Assume `array` is an $M \times N$ matrix ($rows \times cols$) and that $M$ and $N$ are both large.

2.1

```
public static int redHerring(int[][] array) {
    if (array.length < 1 || array[0].length <= 4) {
        return 0;
    }
    for (int i = 0; i < array.length; i++) {
         for (int j = 0; j < array[i].length; j++) {
            if (j == 4) {
                return -1;
            }
        }
    }
    return 1;
}
```

$\Theta(1)$. The function will return once it reaches the fourth element of the first row of the `array` matrix, so this function always takes constant time. (And if the array doesn't have any rows, or if the first row has fewer than 4 entries, the function returns immediately.)

2.2

```
public static int crimsonTuna(int[][] array) {
    if (array.length < 4) {
        return 0;
    }
    for (int i = 0; i < array.length; i++) {
        for (int j = 0; j < array[i].length; j++) {
              if (i == 4) {
                return -1;
            }
        }
    }
    return 1;
}
```

$\Theta(N)$. This function returns after it reaches the fourth row (and if there are less than four rows it returns immediately). The number of elements it's able to reach in four rows is dependent on the number of columns, which in this case is $N$.

Note: If we did not specify that M and N were both large, we would need to give an $O$ bound instead of a $\Theta$ bound here. The reason is that we would need to consider all three cases where the size of the input gets large:

1. $M$ approaches infinity, $N$ is small
2. $N$ approaches infinity, $M$ is small
3. $M$ and $N$ both approach infinity

If all three cases have the same $\Omega$ and $O$ bound then we can write a $\Theta$ bound for this problem. However, if we consider case 2, we can see that if $M < 4$, then the function will return immediately after the first if statement. Thus not all three cases have the same $\Omega$ and $O$ bound, so we can only provide an $O$ bound for the runtime.

2.3

```
public static int pinkTrout(int a) {
    if (a % 7 == 0) {
        return 1;
    } else {
        return pinkTrout(a - 1) + 1;
    }
}
```

$\Theta(1)$. If $a$ is a multiple of 7, this function will return immediately. If not, it will call `pinkTrout(a - 1)` and add one to its result. We know that, at most, it will take 6 calls until `pinkTrout` is called on a multiple of 7, so thus the function takes constant time.

2.4 (a) Give a $O(\cdot)$ runtime bound as a function of $N$, `sortedArray.length`.

```
private static boolean scarletKoi(int[] sortedArray, int x, int start, int end) {
    if (start == end || start == end - 1) {
        return sortedArray[start] == x;
    }
    int mid = end + ((start - end) / 2);
    return sortedArray[mid] == x ||
           scarletKoi(sortedArray, x, start, mid) ||
           scarletKoi(sortedArray, x, mid, end);
}
```

$O(N)$

This method is a trap, as it seems like a binary search. But in the recursive case, we make recursive calls on both the left and right sides, *without taking advantage of the sorted array*. We can craft an input that requires exploring the entire array in linear time.

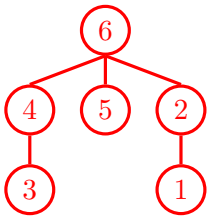
(b) Why can we only give a $O(\cdot)$ runtime and not a $\Theta(\cdot)$ runtime?

In the best case, the middle element will be equal to `x`, in which case the runtime will be $\Theta(1)$. Thus, overall, the runtime of the function is in $\Omega(1), O(N)$ and there's no $\Theta(\cdot)$ runtime because the $\Omega(\cdot)$ and $O(\cdot)$ runtimes don't match.

# 3 Disjoint Sets

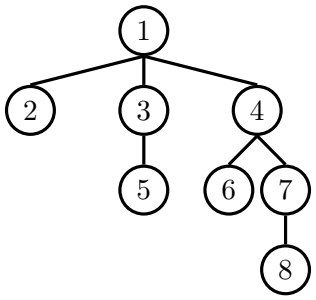
3.1 Suppose we have a `WeightedQuickUnionUF` disjoint set with path compression. Show the tree structure in the union-find algorithm as the following sequence of commands is executed.

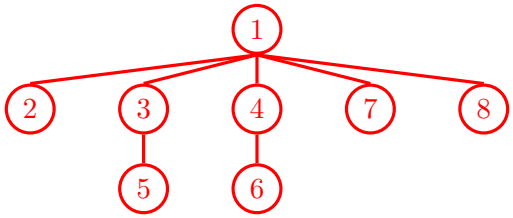
```
connect(1, 2);
connect(3, 4);
connect(5, 6);
connect(1, 6);
connect(3, 6);
```




3.2 Suppose we have the `WeightedQuickUnionUF` tree with path compression below. Draw the tree after we call:

```
find(8);
```




find(8) attempts to find the parent of 8 by searching up the tree through 7, 4, and 1, returning 1. Path compression reassigns the immediate parent of 8, 7, and 4 to 1 (note: that 4's immediate parent doesn't change).

3.3 Describe how to construct a `WeightedQuickUnionUF` tree of maximum height.

To construct a max-height tree of $N$ nodes, union two max-height trees with $\frac{N}{2}$ nodes each. The height of the final tree is 1 greater than each of the max-height $\frac{N}{2}$-node trees.