CSM 61B Spring 2019

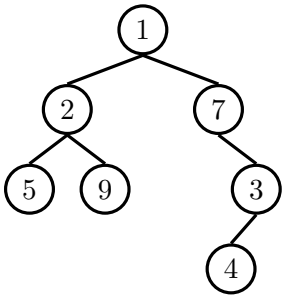
Graphs & Searches

Mentoring 9: April 1, 2019

1 Tree Traversal

Level-Order Traversals Nodes are visited top-to-bottom, left-to-right.

Depth-First Traversals Visit deep nodes before shallow ones.



1.1 Give the ordering for each depth-first traversal of the tree.

(a) Pre-order

1 – 2 – 5 – 9 – 7 – 3 – 4

(b) In-order

5 – 2 – 9 – 1 – 7 – 4 – 3

(c) Post-order

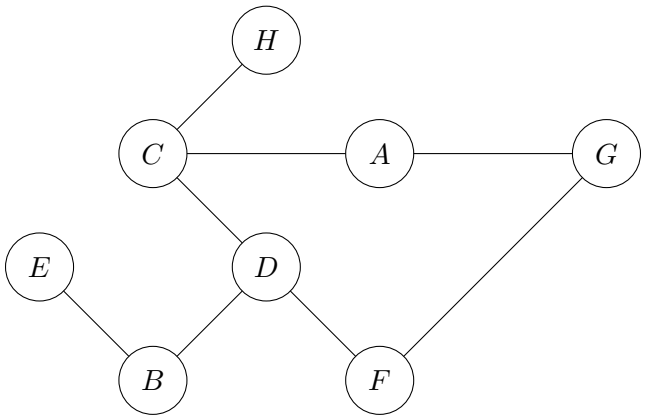
5 – 9 – 2 – 4 – 3 – 7 – 1

1.2 Give the level-order traversal of the tree.

1 – 2 – 7 – 5 – 9 – 3 – 4

2 Searches

2.1 For the graph below, write the order in which vertices are visited using the specified algorithm starting from A. Break ties by alphabetical order.



(a) DFS

$A - C - D - B - E - F - G - H$

Use a Stack when traversing through this graph, similar to tree traversal.

(b) BFS

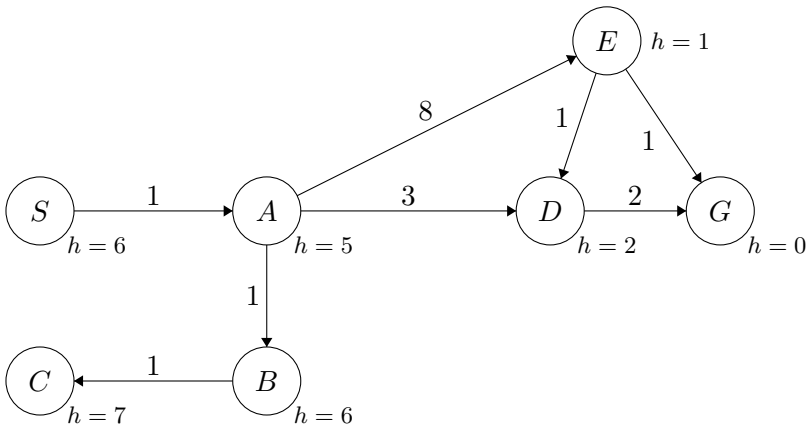
$A - C - G - D - H - F - B - E$

Use a Queue when traversing through the graph, similar to tree traversal.

3 Shortest Paths

3.1 Find the path from the start, S, to the goal, G, when running each of the following algorithms.

The **heuristic**, h, estimates the distance from each node to the goal.



(a) Which path does Dijkstra's return?

$S - A - D - G$

From the starting node, choose the path that has the least cost, go to that node, and repeat until we reach the goal node. We choose the lowest *total cost*.

We keep a priority-queue fringe that keeps track of paths. At each step, we remove the shortest path from the fringe and add its children to the fringe, trying all paths in increasing cost order until we reach G.

(b) Which path does A* search return?

A* search is an algorithm that combines the total distance from the start with the heuristic to optimize the search procedure.

$S - A - D - G$

At each node, we choose the next node that has the lowest sum of the path cost and $h(\cdot)$ value. This is essentially uniform cost search and greedy search combined.

For A* to work, heuristics must be *admissible* and *consistent*.

- Admissible heuristics underestimate the true distance to the goal.
- Consistent heuristics require that the difference in heuristic values between two nodes cannot be greater than the true distance between the two.

(c) What is the runtime of Dijkstra's? A*? What is the space requirement

for both?

We assume a binary heap Priority Queue. The largest the PQ can ever get is size V since there are V vertices, and we never add vertices twice. (We update using `decrementKey` instead.)

In the worst case, we do the following in Dijkstra's:

- Insert every vertex into the PQ (V vertexes, $O(\log V)$ time)
- Remove every vertex from the PQ (V vertexes, $O(\log V)$ time)
- Update every vertex in the PQ (E edges, $O(\log V)$ time)

Hence, Dijkstra's has runtime $O(V \log V + V \log V + E \log V) = O(E \log V)$ since $E > V$.

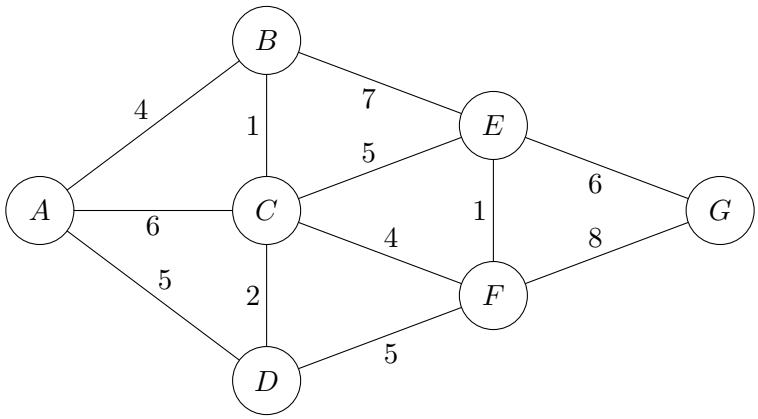
A* has the same runtime as Dijkstra's in the worst case. We can see this by constructing a very poor heuristic that returns 0 for all vertices! We can see that this heuristic is trivially admissible (distance to the goal is at least 0, so it must be admisible) and trivially consistent (the difference is always 0, which is not greater than the true distance between any two nodes). Then, the behaviour of A* on the graph is exactly like Dijkstra's.

However, given a good heuristic, A* can have a better average runtime, which is why we often prefer it.

The space requirement for the graph is $\Theta(V+E)$ assuming an adjacency list, and the space for the priority queue is $\Theta(V)$;

4 Networking

4.1 Suppose we want to design a telephone network connecting all the cities, labeled A to G, in a neighborhood. We'd like to do so at the least cost.



(a) In a graph with N vertices and M edges, how many edges form a minimum spanning tree?

$N - 1$, or 6 edges in the above graph.

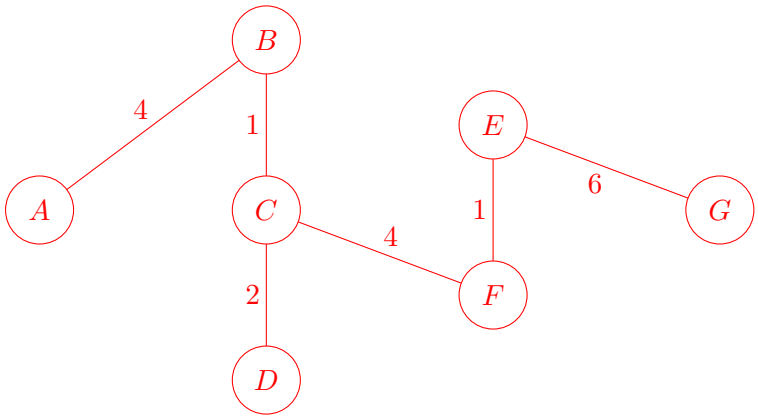
(b) Will the new graph contain any cycles? Describe its structure.

The resulting graph is a tree which implies that it contains no cycles. If the tree reaches every node in the graph, then it is a **spanning tree**. A graph may have many spanning trees, but we are particularly interested in **minimum spanning trees**, or spanning trees that minimize the total weight of the tree.

(c) One algorithm to find a minimum spanning tree is Kruskal's algorithm.

1. Sort all the edges by increasing order of their weight.
2. Pick the smallest edge and check if it forms a cycle with the spanning tree so far. If it doesn't form a cycle, add this edge to the spanning tree.
3. Repeat the previous step until there are $|V| - 1$ edges in the spanning tree, where $|V|$ is the number of vertices in the graph.

Run Kruskal's Algorithm to find a minimum spanning tree.



Meta: Draw only the nodes and add the edges that are part of the MST as you walk through the problem. Labeling all the edges would take a long time, and erasing is confusing and complicated.

5 Algorithms *Extra Practice*

Traversal Visit all the nodes in the graph.

- Depth-first traversal (preorder and postorder)
- Level-order traversal

Search Given s, find a goal v.

- Depth-first search
- Iterative-deepening depth-first search
- Breadth-first search

Single Pair Shortest Path Given s, find the shortest path to a goal v.

- *Uniform cost search*
- *Greedy search*
- A* search

Single Source Shortest Path Given s, find the shortest path to all nodes.

- Dijkstra's algorithm

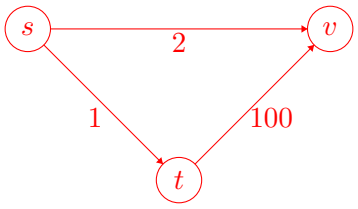
Minimum Spanning Tree A *spanning tree*, or acyclic subgraph connecting all the nodes with the least total edge weight.

- Prim's algorithm
- Kruskal's algorithm

5.1 Is this algorithm for computing the *single pair shortest path* correct?

Given a starting vertex, s, and an ending vertex, v, compute the shortest path between s and v by running DFS, but at each node exploring the shortest outgoing edge first until v is reached. Return the s to v path in the DFS tree.

Incorrect. Consider the graph below whose true shortest path is $s - v$ but whose greedy DFS path is $s - t - v$.



5.2 Briefly describe an efficient algorithm and the runtime for finding a minimum spanning tree in an undirected, connected graph $G = (V, E)$ when the edge weights satisfy:

(a) For all $e \in E$, $w_e = 1$. (All edge weights are 1.)

The key idea here is that any tree which connects all nodes is an MST. We can run DFS and take the DFS tree. You could also take a BFS tree, or run Prim's algorithm with a queue or stack instead of a priority queue (this would be equivalent to BFS/DFS). The runtime of this algorithm is in $\Theta(|V| + |E|) \in \Theta(|E|)$ for simple graphs.

(b) For all $e \in E$, $w_e \in \{1, 2\}$. (All edge weights are either 1 or 2.)

Run Prim's algorithm with a specialized priority queue, comprised of 2 regular queues. When we add items to the priority queue, we add it to the first queue if the weight is 1 and otherwise add it to the second queue (the weight must be 2). When we pop from the priority queue, we take from the first queue, unless it is empty, in which case we take from the second. The runtime of this algorithm is in $\Theta(|E|)$ as priority queue operations are now constant time.

5.3 Given a weighted, directed graph G where the weights of every edge in G are all integers between 1 and 10, and a starting vertex s in G, find the distance from s to every other vertex in the graph where the distance between two vertices is defined as the weight of the shortest path connecting them, or infinity if no such path exists.

(a) Design an algorithm for solving the problem better than Dijkstra's.

For every edge e in the graph, replace e with a chain of $w - 1$ vertices (where w is the weight of e) where the two ends of the chain are the endpoints of e.

Then run BFS on the modified graph, keeping track of the distance from v to each vertex from the original graph.

Alternatively, we can modify Dijkstra's algorithm. Since the runtime of Dijkstra's is bounded by the priority queue implementation, if we can come up with a faster priority queue, we can improve the runtime. Define our priority queue as an array of 11 linked list buckets. Keep track of a counter that represents our current position in the array. Each bucket corresponds to vertices of some distance from the start, s.

`removeMin()`: If the bucket with index counter is non-empty, remove and return the first vertex in its linked list. Otherwise, increment counter until we find a non-empty bucket. If the counter reaches 11, reset it to 0 and continue (so in effect our array is circular).

`insert()`: Given a vertex v and a distance d, insert the vertex into the beginning of the linked list at index $d \mod 11$.

This strategy works because the vertices we add to the priority queue at any time have a distance that is no more than 10 greater than the current distance.

(b) Give the runtime of your algorithm.

$\Theta(|V| + |E|)$ for running BFS on the modified graph, and $O(|V| + |E|)$ for modifying Dijkstra's priority queue.